Finite-Dimensional Hamiltonian Systems from Li Spectral Problem by Symmetry Constraints

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A hierarchy associated with the Li spectral problem is derived with the help of the zero curvature equation. It is shown that the hierarchy possesses bi-Hamiltonian structure and is integrable in the Liouville sense. Moreover, the mono- and binary-nonlinearization theory can be successfully applied in the spectral problem. Under the Bargmann symmetry constraints, Lax pairs and adjoint Lax pairs are nonlineared into finite-dimensional Hamiltonian systems (FDHS) in the Liouville sense. New involutive solutions for the Li hierarchy are obtained.

Key words: Li Spectral Problem; Symmetry Constraint; Hamiltonian System; Involutive Solution.

1. Introduction

A central and difficult topic in the study of integrable systems is to find Liouville integrable systems such that those associate with certain nonlinear evolution equations taking physical significance. Nonlinearization technique or symmetry constraints are proved to be a powerful tool for obtaining new finite-dimensional integrable Hamiltonian systems [1]. Under Bargmann or Neumann constraints between the potentials and the eigenfunctions which play a central role in the process of nonlinearization, the spectral problem is nonlinearized as a finite-dimensional completely integrable systems. The list covers the spectral problems associated with the well-known soliton hierarchies such as AKNS, JM, KN [2 – 4]. In recent years, a binary-nonlinearization technique of Lax pairs has been developed [3]. This method can be applied in matrix spectral problems with any order.

In this paper, we consider the Li spectral problem

$$\varphi_x = U(u,v,\lambda)\varphi = \begin{pmatrix} -\lambda+v & u+v \\ u-v & \lambda-v \end{pmatrix}\varphi, \quad (1.1)$$

where $\varphi = (\varphi_1,\varphi_2)^T$, λ is a constant spectral parameter, and u, v are two potentials. This spectral problem was first presented by Li and Chen [5]. Starting from the spectral problem (1.1), the Li hierarchy was given in [6]. Here we are interested in the nonlinearization of spectral problem (1.1) to find finite-dimensional Hamiltonian systems (FDHS). In the following section, we first recall the Li hierarchy for the need of constraints. In Section 3, we study the mono-nonlinearization of the Li spectral problem (1.1) to find FDHS. In Section 4, we further analyze the binary-nonlinearization of the spectral problem and adjoint a spectral problem under the Bargmann symmetry constraint. A complete integrable Hamiltonian system is then established. We then present a kind of involutive solutions to the Li hierarchy in Section 5.

2. Li Spectral Problem and its Hierarchy

Starting from spectral problem (1.1), the Li hierarchy and its Hamilton structure were obtained by using the zero curvature equation and trace identity [7]. To find the symmetry constraint and useful formulae, we recall the Li hierarchy here.

Solving the stationary zero-curvature equation

$$V_x = [U,V], \quad V = \begin{pmatrix} a+c & b+c \\ b-c & -a-c \end{pmatrix} \quad (2.1)$$

with

$$a = \sum_{m\geq 0} a_m\lambda^{-m}, \quad b = \sum_{m\geq 0} b_m\lambda^{-m}, \quad c = \sum_{m\geq 0} c_m\lambda^{-m},$$

we obtain the following recurrence relations:

$$a_0 = \alpha, \quad b_0 = c_0 = 0,$$

$$a_{mx} = 2ua_m + 2b_{m+1},$$
$$b_{mx} = -2va_m - 2c_{m+1},$$
$$c_{mx} = -2ua_m + 2vb_m - 2uc_m - 2b_{m+1},$$
$$\begin{pmatrix} b_n \\ a_n \end{pmatrix} = L \begin{pmatrix} b_{n-1} \\ a_{n-1} \end{pmatrix}, \tag{2.2}$$

where

$$L = \begin{pmatrix} 0 & \frac{1}{2}\partial - u \\ \frac{1}{2}\partial + \partial^{-1}u\partial & v + \partial^{-1}v\partial \end{pmatrix}.$$

The first few quantities are given by

$$\begin{aligned} &a_1 = \alpha v, \quad b_1 = -\alpha u, \quad c_1 = -\alpha v, \\ &a_2 = -\alpha/2(u_x + u^2 - 3v^2), \\ &b_2 = \alpha/2(v_x - 2uv), \quad c_2 = \alpha/2(u_x - 2v^2). \end{aligned} \tag{2.3}$$

Introducing the auxiliary spectral problem

$$\varphi_{t_n} = V^{(n)}\varphi, \tag{2.4}$$

where

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n, \quad \Delta_n = \begin{pmatrix} -a_n & 0 \\ 0 & a_n \end{pmatrix},$$

the compatibility conditions between (1.1) and (2.4) lead to the Li hierarchy

$$\begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix} = \begin{pmatrix} b_n \\ a_n \end{pmatrix} = JL \begin{pmatrix} b_{n-1} \\ a_{n-1} \end{pmatrix} = \alpha JL^{n-1} \begin{pmatrix} -u \\ v \end{pmatrix}, \tag{2.5}$$

where

$$J = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix},$$

$$K = JL = \begin{pmatrix} 0 & \frac{1}{2}\partial^2 - \partial u \\ -\frac{1}{2}\partial^2 - u\partial & -\partial v - v\partial \end{pmatrix},$$

and J, K are two Hamiltonian operators.

By using trace identity, we can show that the hierarchy (2.5) is integrable in the Liouville sense and possesses bi-Hamiltonian structure:

$$\begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta w} = K\frac{\delta H_n}{\delta w}, \tag{2.6}$$

where Hamiltonian functions are given by

$$H_1 = \alpha v, \quad H_{n+1} = \frac{1}{n}(a_{n+1} + c_{n+1}), \quad w = (u,v)^T.$$

The first two nontrivial equations in the Li hierarchy (2.5) are

$$u_{t_1} = -\alpha u_x, \quad v_{t_1} = -\alpha v_x,$$
$$u_{t_2} = \frac{1}{2}\alpha(vx - 2uv)_x, \quad v_{t_2} = \frac{1}{2}\alpha(u_x + u^2 - 3v^2)_x.$$

Particularly, when $u = v$, $\alpha = 2$, the second equations are the reduced Burgers equation

$$u_t = u_{xx} - 4uu_x.$$

3. Mono-Nonlinearization of the Spectral Problem

Let $\lambda_1, \lambda_2, \ldots \lambda_N$ be N distinct real numbers and $\varphi = (p_j, q_j)^T$ corresponding eigenvalues λ_j, we take N copies of the spectral problem (1.1):

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = U(u,v,\lambda_j)\begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad j = 1,\ldots,N. \tag{3.1}$$

By computing the variation derivative of λ, we have

$$\begin{aligned} \nabla\lambda_j &= (\delta\lambda_j/\delta u, \delta\lambda_j/\delta v)^T \\ &= c(-p_j^2 + q_j^2, 2p_jq_j + p_j^2 + q_j^2), \end{aligned} \tag{3.2}$$

where $c = (2\int_{-\infty}^{\infty} p_jq_j \mathrm{d}x)^{-1}$. We consider the Bargmann symmetry constraint

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \Sigma_{j=1}^N \frac{\alpha}{c}\nabla\lambda_j, \tag{3.3}$$

which leads to

$$\begin{aligned} u &= \langle p,p\rangle - \langle q,q\rangle, \\ v &= 2\langle p,q\rangle + \langle p,p\rangle + \langle q,q\rangle, \end{aligned} \tag{3.4}$$

where $p = (p_1,\ldots,p_N)^T$, $q = (q_1,\ldots,q_N)^T$, and $\langle .,. \rangle$ stands for the standard inner product in R^N.

Substituting (3.4) into (3.1) generates the following FDHS:

$$\begin{aligned} p_x &= -\wedge p + (2\langle p,q\rangle + \langle p,p\rangle + \langle q,q\rangle)p \\ &\quad + 2(\langle p,p\rangle + \langle p,q\rangle)q = \frac{\partial H}{\partial q}, \\ q_x &= -2(\langle q,q\rangle + \langle p,q\rangle)p + \wedge q \\ &\quad - (2\langle p,q\rangle + \langle p,p\rangle + \langle q,q\rangle)q = -\frac{\partial H}{\partial p}, \end{aligned} \tag{3.5}$$

where $\wedge = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and

$$H = -\langle \wedge p, q\rangle + (\langle p,q\rangle + \langle p,p\rangle + \langle q,q\rangle)\langle p,q\rangle + \langle p,p\rangle\langle q,q\rangle. \tag{3.6}$$

Making use of the relation

$$K\nabla\lambda_j = \lambda_j J\nabla\lambda_j, \tag{3.7}$$

we have

$$\begin{aligned}
&\tilde{a}_i = a_i, \quad \tilde{b}_i = b_i, \quad \tilde{c}_i = c_i, \quad i = 0,1,\\
&\tilde{a}_m = \alpha(2\langle \wedge^{m-1} p,q\rangle + \langle \wedge^{m-1} p,p\rangle + \langle \wedge^{m-1} q,q\rangle), \quad m \ge 2,\\
&\tilde{b}_m = \alpha(\langle \wedge^{m-1} q,q\rangle - \langle \wedge^{m-1} p,p\rangle), \quad m \ge 2,\\
&\tilde{c}_m = -\alpha(\langle \wedge^{m-1} q,q\rangle + \langle \wedge^{m-1} p,p\rangle), \quad m \ge 2.
\end{aligned} \tag{3.8}$$

Under the conditions (3.4) and (3.8), it is easy to verify the following propositions.

Proposition 1. The Bargmann system (3.5) enjoys the Lax equation

$$\tilde{V}_x = [\tilde{U}, \tilde{V}], \tag{3.9}$$

where the tilde denotes the corresponding expression under the conditions (3.4) and (3.8).

Let

$$F = \det\tilde{V} = \sum_{m=0}^{\infty} F_m \lambda^{-m},$$

we have

$$F_0 = \alpha^2, \quad F_1 = 0,$$

$$F_2 = 4\alpha^2[\langle \wedge p,q\rangle - (\langle p,q\rangle + \langle p,p\rangle)(\langle q,q\rangle + \langle p,q\rangle)],$$

$$F_3 = 4\alpha^2[\langle \wedge^2 p,q\rangle - (\langle p,p\rangle + \langle p,q\rangle)\langle \wedge q,q\rangle - (\langle p,q\rangle + \langle q,q\rangle)\langle \wedge p,p\rangle],$$

$$\begin{aligned}
F_m = 4\alpha^2\Big\{&[\langle \wedge^{m-1} p,q\rangle - (\langle p,p\rangle + \langle p,q\rangle)\langle \wedge^{m-2} q,q\rangle - (\langle p,q\rangle + \langle q,q\rangle)\langle \wedge^{m-2} p,p\rangle]\\
&+ \sum_{i=2}^{m-2}[\langle \wedge^{i-1} p,q\rangle\langle \wedge^{m-i-1} p,q\rangle - \langle \wedge^{i-1} p,p\rangle\langle \wedge^{m-i-1} q,q\rangle]\Big\},
\end{aligned}$$

$$m \ge 4. \tag{3.10}$$

Specifically

$$F_2 = -4\alpha^2 H.$$

From (3.9), it is easy to verify that

$$D_x F_m = 0, \quad m \ge 0,$$

which means F_m constitutes a hierarchy of integrals of motion for FDHS (3.5).

In the same way, we can discuss the nonlinearization of the auxiliary spectral problem

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_{t_n} = \tilde{V}^{(n)}(u,v,\lambda_j)\begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad j = 1,\ldots,N. \tag{3.11}$$

Under the constraints (3.4) and (3.8), we obtain a hierarchy of finite-dimensional systems with the Bargmann symmetry constraints for (3.11); we call them restricted t_n flow. It is easy to show the identity

$$\tilde{V}_{t_n} = [\tilde{V}^{(n)}, \tilde{V}]. \tag{3.12}$$

This implies that F_m also constitutes a hierarchy of integrals of motion for system (3.11).

Through a direct calculation, we find that system (3.11) can be rewritten as

$$p_{t_n} = \frac{\partial(\frac{1}{4\alpha}F_{n+1})}{\partial q}, \; q_{t_n} = -\frac{\partial(\frac{1}{4\alpha}F_{n+1})}{\partial p}, \; n \ge 1. \tag{3.13}$$

We consider the standard symplectic structure on R^{2N}, then the poisson bracket for two smooth functions f and g in the symplectic space $(R^{2N}, dp \wedge dq)$ is defined as

$$\{f,g\} = \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right).$$

Proposition 2. F_2, F_3, $\ldots F_{N+1}$ are involutions in pairs.

Proof: A direct calculation gives

$$\{F_m, F_n\} = -4\alpha\frac{\mathrm{d}F_m}{\mathrm{d}t_n}.$$

From the Lax presentation (3.12) we get

$$\frac{\mathrm{d}F_m}{\mathrm{d}t_n} = 0,$$

which implies $\{F_m, F_n\} = 0$. So the proposition is proved.

Proposition 3. F_2, F_3, $\ldots F_{N+1}$ are functionally independent over some region of R^{2N}.

Proof: Direct computation leads to

$$\frac{\partial F_l}{\partial p}|_{p=0} = 4\alpha^2 \wedge^{l-1} q, \quad l = 2\ldots N+1.$$

Therefore we obtain

$$\det \begin{pmatrix} \frac{\partial F_2}{\partial p_1} & \frac{\partial F_3}{\partial p_1} & \cdots & \frac{\partial F_{N+1}}{\partial p_1} \\ \frac{\partial F_2}{\partial p_2} & \frac{\partial F_3}{\partial p_2} & \cdots & \frac{\partial F_{N+1}}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_2}{\partial p_N} & \frac{\partial F_3}{\partial p_N} & \cdots & \frac{\partial F_{N+1}}{\partial p_N} \end{pmatrix} |_{p=0}$$

$$= (4\alpha^2)^N \det \begin{pmatrix} \lambda_1 q_1 & \lambda_1^2 q_1 & \ldots & \lambda_1^N q_1 \\ \lambda_2 q_2 & \lambda_2^2 q_2 & \ldots & \lambda_2^N q_2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_N q_N & \lambda_N^2 q_N & \ldots & \lambda_N^N q_N \end{pmatrix}$$

$$= (4\alpha^2)^N q_1 \ldots q_N \Pi_{i\neq j}(\lambda_i - \lambda_j).$$

This means that functions F_2, F_3, $\ldots F_{N+1}$ can be functionally independent at least over some region of R^{2N}.

From propositions 2 and 3, we immediately obtain the following theorem.

Theorem 1. Both the finite-dimensional Hamiltonian systems (3.5) and (3.13) are completely integrable systems in the Liouville sense. Besides, the x flow and t_n flow are commutative in the symplectic space (R^{2N}, $\mathrm{d}p \wedge \mathrm{d}q$).

4. Binary-Nonlinearization of Spectral Problem

In the following, we would like to perform binary-nonlinearization for the Lax pairs and the adjoint Lax pairs. We consider the adjoint spectral problem of (1.1)

$$\psi_x = -U(u,v,\lambda)^T \psi, \quad \psi = (\psi_1, \psi_2)^T \qquad (4.1)$$

and its auxiliary spectral problem

$$\psi_{t_n} = -V^{(n)}(u,v,\lambda)^T \psi. \qquad (4.2)$$

Similarly, let λ_1, λ_2, $\ldots \lambda_N$ be N distinct real numbers and $\psi = (\psi_{1j}, \psi_{2j})^T$ corresponding eigenvalues λ_j, we take N copies of the adjoint Lax pairs of (4.1) and (4.2):

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = -U(u,v,\lambda_j)^T \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1,\ldots,N, \qquad (4.3)$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -V^n(\lambda_j)^T \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1,\ldots,N. \qquad (4.4)$$

In order to impose the Bargmann symmetry constraint in binary-nonlinearization, by means of the formulae in [8], we get (up to a constant factor)

$$\frac{\delta \lambda_j}{\delta u} = -\alpha(\varphi_{1j}\psi_{2j} + \varphi_{2j}\psi_{1j}), \qquad (4.5)$$

$$\frac{\delta \lambda_j}{\delta v} = -\alpha(\varphi_{1j}\psi_{1j} - \varphi_{1j}\psi_{2j} + \varphi_{2j}\psi_{1j} - \varphi_{2j}\psi_{2j}).$$

Now we make the Bargmann symmetry constraint

$$J\frac{\delta H_2}{\delta w} = J\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta w}, \qquad (4.6)$$

from which we get the following explicit expression for the potentials:

$$u = \langle \Phi_1, \Psi_2 \rangle + \langle \Phi_2, \Psi_1 \rangle, \qquad (4.7)$$

$$v = -\langle \Phi_1, \Psi_1 \rangle + \langle \Phi_1, \Psi_2 \rangle - \langle \Phi_2, \Psi_1 \rangle + \langle \Phi_2, \Psi_2 \rangle,$$

where we denote the inner product in R^N by $\langle .,. \rangle$, and $\Phi_i = (\varphi_{i1}, \ldots, \varphi_{iN})^T$, $\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^T$, $i = 1,2$, $\wedge = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.

Under the constraint (4.7), the spatial parts (3.1) and (4.3) of the Lax pairs and the adjoint Lax pairs can be rewritten as

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_x = U(u,v,\lambda_j)|_B \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = -U(u,v,\lambda_j)^T|_B \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1,\ldots,N,$$

where the subscript B means substitution of (4.7) into the expressions.

The above system can be expressed as the FDHS

$$\frac{\partial \Phi_i}{\partial x} = \frac{\partial \tilde{H}}{\partial \Psi_i}, \quad \frac{\partial \Psi_i}{\partial x} = -\frac{\partial \tilde{H}}{\partial \Phi_i}, \quad i = 1,2, \qquad (4.8)$$

and its Hamiltonian function is

$$\begin{aligned} \tilde{H} = & -\langle \wedge \Phi_1, \Psi_1 \rangle + \langle \wedge \Phi_2, \Psi_2 \rangle \\ & -\frac{1}{2}\langle \Phi_1, \Psi_1 \rangle^2 - \frac{1}{2}\langle \Phi_2, \Psi_2 \rangle^2 \\ & + (2\langle \Phi_1, \Psi_2 \rangle + \langle \Phi_2, \Psi_2 \rangle)\langle \Phi_2, \Psi_1 \rangle \\ & + (\langle \Phi_1, \Psi_2 \rangle + \langle \Phi_2, \Psi_2 \rangle)\langle \Phi_1, \Psi_1 \rangle \\ & - \langle \Phi_1, \Psi_1 \rangle\langle \Phi_2, \Psi_1 \rangle - \langle \Phi_2, \Psi_2 \rangle\langle \Phi_1, \Psi_2 \rangle. \end{aligned}$$

By direct calculation, we have

$$K\frac{\delta\lambda_j}{\delta w} = \lambda_j J \frac{\delta\lambda_j}{\delta w}, \tag{4.9}$$

from which we get

$$\begin{aligned}
&\tilde{a}_0 = \alpha, \quad \tilde{b}_0 = \tilde{c}_0 = 0,\\
&\tilde{a}_1 = \alpha(-\langle\Phi_1,\Psi_1\rangle + \langle\Phi_1,\Psi_2\rangle - \langle\Phi_2,\Psi_1\rangle + \langle\Phi_2,\Psi_2\rangle),\\
&\tilde{b}_1 = -\alpha(\langle\Phi_1,\Psi_2\rangle + \langle\Phi_2,\Psi_1\rangle),\\
&\tilde{c}_1 = -\alpha(-\langle\Phi_1,\Psi_1\rangle + \langle\Phi_1,\Psi_2\rangle - \langle\Phi_2,\Psi_1\rangle + \langle\Phi_2,\Psi_2\rangle),\\
&\tilde{a}_m = -\alpha\big(\langle\wedge^{m-1}\Phi_1,\Psi_1\rangle - \langle\wedge^{m-1}\Phi_1,\Psi_2\rangle\\
&\qquad + \langle\wedge^{m-1}\Phi_2,\Psi_1\rangle - \langle\wedge^{m-1}\Phi_2,\Psi_2\rangle\big),\\
&\tilde{b}_m = -\alpha(\langle\wedge^{m-1}\Phi_1,\Psi_2\rangle + \langle\wedge^{m-1}\Phi_2,\Psi_1\rangle), \quad m \geq 2,\\
&\tilde{c}_m = -\alpha(\langle\wedge^{m-1}\Phi_1,\Psi_2\rangle - \langle\wedge^{m-1}\Phi_2,\Psi_1\rangle),
\end{aligned}$$

and from [9] we have

$$(V|_B)_x = [U|_B, V|_B], \quad (V|_B)_{t_n} = [V^{(n)}|_B, V|_B]. \tag{4.10}$$

Then

$$\begin{aligned}
\det(V|_B) = &\Big(\sum_{m\geq 0}(\tilde{a}_m + \tilde{c}_m)\lambda^{-m}\Big)^2\\
&-\Big(\sum_{m\geq 0}(\tilde{b}_m + \tilde{c}_m)\lambda^{-m}\Big)\Big(\sum_{m\geq 0}(\tilde{b}_m - \tilde{c}_m)\lambda^{-m}\Big).
\end{aligned}$$

Setting

$$\det(V|_B) = \tilde{F} = \sum_{m\geq 0}\tilde{F}_m\lambda^{-m},$$

we have

$$\begin{aligned}
&\tilde{F}_0 = \alpha^2, \quad \tilde{F}_1 = 0,\\
&\tilde{F}_2 = \alpha^2\big\{-2(\langle\wedge\Phi_1,\Psi_1\rangle - \langle\wedge\Phi_2,\Psi_2\rangle)\\
&\qquad + (-\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_1,\Psi_2\rangle + \langle\Phi_2,\Psi_2\rangle)\\
&\qquad \cdot(\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_2,\Psi_1\rangle - \langle\Phi_2,\Psi_2\rangle)\big\},\\
&\tilde{F}_3 = \alpha^2\big\{-2(\langle\wedge^2\Phi_1,\Psi_1\rangle - \langle\wedge^2\Phi_2\Psi_2\rangle)\\
&\qquad + 2(-\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_1,\Psi_2\rangle + \langle\Phi_2,\Psi_2\rangle)\langle\wedge\Phi_2,\Psi_1\rangle\\
&\qquad + 2(\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_2,\Psi_1\rangle - \langle\Phi_2,\Psi_2\rangle)\langle\wedge\Phi_1,\Psi_2\rangle\big\},\\
&\tilde{F}_m = \alpha^2\big\{-2\langle\wedge^{m-1}\Phi_1,\Psi_1\rangle - \langle\wedge^{m-1}\Phi_2,\Psi_2\rangle\\
&\quad + 2(-\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_1,\Psi_2\rangle + \langle\Phi_2,\Psi_2\rangle)\langle\wedge^{m-2}\Phi_2,\Psi_1\rangle\\
&\quad + 2(\langle\Phi_1,\Psi_1\rangle + 2\langle\Phi_2,\Psi_1\rangle - \langle\Phi_2,\Psi_2\rangle)\langle\wedge^{m-2}\Phi_1,\Psi_2\rangle\\
&\quad + \sum_{i=2}^{m-2}(\langle\wedge^{i-1}\Phi_1,\Psi_1\rangle - \langle\wedge^{i-1}\Phi_2,\Psi_2\rangle)\\
&\qquad \cdot(\langle\wedge^{m-i-1}\Phi_1,\Psi_1\rangle - \langle\wedge^{m-i-1}\Phi_2,\Psi_2\rangle)\\
&\quad + 4\sum_{i=2}^{m-2}\langle\wedge^{i-1}\Phi_1,\Psi_2\rangle\cdot\langle\wedge^{m-i-1}\Phi_2,\Psi_1\rangle\big\}, \quad m \geq 4.
\end{aligned}$$

Moreover we notice that $\tilde{H} = \frac{1}{2\alpha^2}\tilde{F}_2$.

The temporal parts of the nonlinearized Lax pairs and adjoint Lax pairs

$$\begin{aligned}
&\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}_{t_n} = V^{(n)}(u,v,\lambda_j)|_B\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix},\\
&\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_n} = -V^{(n)}(u,v,\lambda_j)^T|_B\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix},\\
&j = 1,\dots,N
\end{aligned}$$

also can be expressed in the Hamiltonian form:

$$\frac{\partial\Phi_i}{\partial t_n} = \frac{1}{2\alpha^2}\frac{\partial\tilde{F}_{n+1}}{\partial\Psi_i}, \quad \frac{\partial\Psi_i}{\partial t_n} = -\frac{1}{2\alpha^2}\frac{\partial\tilde{F}_{n+1}}{\partial\Phi_i}, \quad i = 1,2. \tag{4.11}$$

Next we consider the standard symplectic structure on R^{4N}:

$$w^2 = \mathrm{d}\Psi_1\wedge\mathrm{d}\Phi_1 + \mathrm{d}\Psi_2\wedge\mathrm{d}\Phi_2.$$

The corresponding Poisson bracket is defined as

$$\{F,G\} = \sum_{i=1}^{2}\sum_{j=1}^{N}\Big(\frac{\partial F}{\partial\psi_{ij}}\frac{\partial G}{\partial\varphi_{ij}} - \frac{\partial F}{\partial\varphi_{ij}}\frac{\partial G}{\partial\psi_{ij}}\Big).$$

Set $\bar{F}_j = \psi_{1j}\varphi_{1j} + \psi_{2j}\varphi_{2j}$, $j = 1,\dots,N$, and by direct calculation we have

$$\begin{aligned}
&\frac{\mathrm{d}\bar{F}_j}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}\bar{F}_j}{\mathrm{d}t_n} = 0 \quad j = 1,2,\dots,N,\\
&\{\bar{F}_j,\bar{F}_l\} = 0, \quad \{\bar{F}_j, \quad \tilde{F}_m\} = 0,\\
&l = 1,\dots,N, \quad m = 2,\dots,N+1.
\end{aligned} \tag{4.12}$$

Equations (4.10) and (4.12) imply that the functions $\bar{F}_1,\dots,\bar{F}_N$, $\tilde{F}_2,\dots,\tilde{F}_{N+1}$ are integrals of motion of the systems (4.8) and (4.11).

From (4.10) we have

$$\{\tilde{F}_j,\tilde{F}_l\} = \frac{\partial\tilde{F}_j}{\partial t_l} = 0, \quad j,l \geq 2. \tag{4.13}$$

Therefore, according to (4.12) and (4.13), we find:

Proposition 4. The functions $\bar{F}_1,\dots,\bar{F}_N$, $\tilde{F}_2,\dots,\tilde{F}_{N+1}$ are in involution in pairs with respect to the Poisson bracket here.

Proposition 5. $\bar{F}_1,\dots,\bar{F}_N$, $\tilde{F}_2,\dots,\tilde{F}_{N+1}$ are functionally independent over some region of R^{4N}.

Proof: A direct calculation gives rise to

$$\frac{\partial \bar{F}_j}{\partial \varphi_{il}} = \psi_{ij}\delta_{jl}, \quad j,l = 1,2,\ldots,N,$$

$$\frac{\partial \tilde{F}_l}{\partial \Phi_1}|_{\Phi_1=\Phi_2=0} = -2\wedge^{l-1}\Psi_1,$$

$$\frac{\partial \tilde{F}_l}{\partial \Phi_2}|_{\Phi_1=\Phi_2=0} = 2\wedge^{l-1}\Psi_2, \quad l = 2,\ldots,N+1.$$

Therefore we have:

$$\det\begin{pmatrix} \frac{\partial \bar{F}_1}{\partial \Phi_1} & \cdots & \frac{\partial \bar{F}_N}{\partial \Phi_1} & \frac{\partial \tilde{F}_2}{\partial \Phi_1} & \cdots & \frac{\partial \tilde{F}_{N+1}}{\partial \Phi_1} \\ \frac{\partial \bar{F}_1}{\partial \Phi_2} & \cdots & \frac{\partial \bar{F}_N}{\partial \Phi_2} & \frac{\partial \tilde{F}_2}{\partial \Phi_2} & \cdots & \frac{\partial \tilde{F}_{N+1}}{\partial \Phi_2} \end{pmatrix}|_{\Phi_1=\Phi_2=0}$$

$$= \det\begin{pmatrix} \psi_{11} & \ldots & 0 & -2\lambda_1\psi_{11} & \ldots & -2\lambda_1^N\psi_{11} \\ \ldots & & & & & \\ 0 & \ldots & \psi_{1N} & -2\lambda_N\psi_{1N} & \ldots & -2\lambda_N^N\psi_{1N} \\ \psi_{21} & \ldots & 0 & 2\lambda_1\psi_{21} & \ldots & 2\lambda_1^N\psi_{21} \\ \ldots & & & & & \\ 0 & \ldots & \psi_{2N} & 2\lambda_N\psi_{2N} & \ldots & 2\lambda_N^N\psi_{2N} \end{pmatrix}$$

$$= 2^{2N}\Pi_{l=1}^2\Pi_{i,j=1}^N\psi_{lj}\Pi_{k=1}^N\lambda_k\Pi_{i\neq j}(\lambda_i-\lambda_j).$$

This means that functions $\bar{F}_1,\ldots,\bar{F}_N$, $\tilde{F}_2,\ldots,\tilde{F}_{N+1}$ are functionally independent at least over a certain region of R^{4N}.

So we can lead to the following theorem.

Theorem 2. Both the finite-dimensional Hamiltonian systems (4.8) and (4.11) in the symplectic manifold $(R^{4N},\sum_{i=1}^2 \mathrm{d}\Psi_i\wedge\mathrm{d}\Phi_i)$ are completely integrable systems in the Liouville sense.

5. Involutive Solutions

Consider the canonical systems of $\tilde{H}$ flow and $\frac{1}{2\alpha^2}\tilde{F}_{n+1}$ flow, $n\geq 1$, respectively:

$$(\Phi_1^T,\Phi_2^T,\Psi_1^T,\Psi_2^T)_x^T = \left(\left(\frac{\partial\tilde{H}}{\partial\Psi_1}\right)^T,\left(\frac{\partial\tilde{H}}{\partial\Psi_2}\right)^T,\left(-\frac{\partial\tilde{H}}{\partial\Phi_1}\right)^T,\left(-\frac{\partial\tilde{H}}{\partial\Phi_2})^T\right)\right)$$

$$= I\nabla\tilde{H}, \tag{5.1}$$

$$(\Phi_1^T,\Phi_2^T,\Psi_1^T,\Psi_2^T)_{t_n}^T = \frac{1}{2\alpha^2}\Big(\Big(\frac{\partial\tilde{F}_{n+1}}{\partial\Psi_1}\Big)^T, \Big(\frac{\partial\tilde{F}_{n+1}}{\partial\Psi_2}\Big)^T,\Big(-\frac{\partial\tilde{F}_{n+1}}{\partial\Phi_1}\Big)^T,\Big(-\frac{\partial\tilde{F}_{n+1}}{\partial\Phi_2}\Big)^T\Big)\Big)$$

$$= \frac{1}{2\alpha^2}I\nabla\tilde{F}_{n+1}, \tag{5.2}$$

where I_{2N} is the $2N\times 2N$ unit matrix

$$I = \begin{pmatrix} 0 & I_{2N} \\ -I_{2N} & 0 \end{pmatrix}.$$

Let $g_{\tilde{H}}^x$ and $g_{\tilde{F}_{n+1}/(2\alpha^2)}^{t_n}$ be the Hamiltonian phase flows associated with the Hamiltonian systems (5.1) and (5.2), respectively.

Define

$$\begin{pmatrix} \Phi_1(x,t_n) \\ \Phi_2(x,t_n) \\ \Psi_1(x,t_n) \\ \Psi_2(x,t_n) \end{pmatrix} = g_{\tilde{H}}^x g_{\frac{1}{2\alpha^2}\tilde{F}_{n+1}}^{t_n} \begin{pmatrix} \Phi_1(0,0) \\ \Phi_2(0,0) \\ \Psi_1(0,0) \\ \Psi_2(0,0) \end{pmatrix} \tag{5.3}$$

with $\Phi_i(0,0)$, $\Psi_i(0,0)$, $i=1,2$ being arbitrary constant vectors. Since $\tilde{H}$ and $\frac{1}{2\alpha^2}\tilde{F}_{n+1}$ are in involution, we arrive at the following proposition:

Proposition 6.

(1) The canonical systems (5.1) and (5.2) are compatible.

(2) The Hamiltonian phase flows $\tilde{H}$ and $\frac{1}{2\alpha^2}\tilde{F}_{n+1}$ commute.

Due to the commutativity of $\tilde{H}$ and $\frac{1}{2\alpha^2}\tilde{F}_{n+1}$, (5.3) is the involutive solution of the consistent systems (5.1) and (5.2). The n-th order Li equation (2.5) is the compatibility condition between (1.1) and (2.4). So under the constraint (4.6), (2.5) becomes the compatibility condition between (5.1) and (5.2). Furthermore, (5.3) is the common solution to (5.1) and (5.2). Hence, we have the following theorem.

Theorem 3. Suppose $(\Phi_1(x,t_n)^T$, $\Phi_2(x,t_n)^T$, $\Psi_1(x,t_n)^T$, $\Psi_2(x,t_n)^T)^T$ are involutive solutions of the consistent Hamiltonian systems (5.1) and (5.2), then

$$w = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \langle\Phi_1,\Psi_2\rangle + \langle\Phi_2,\Psi_1\rangle \\ -\langle\Phi_1,\Psi_1\rangle + \langle\Phi_1,\Psi_2\rangle - \langle\Phi_2,\Psi_1\rangle + \langle\Phi_2,\Psi_2\rangle \end{pmatrix} \tag{5.4}$$

satisfy the n-th Li equation (2.5).

According to Theorem 3, Li equation (2.5) are split into finding involutive solutions $\Phi_i(x,t_n)$ and $\Psi_i(x,t_n)$, $i=1,2$. This kind of involutive representation of solutions to integrable systems exhibits both the interrelation between (1+1)-dimensional integrable systems and finite-dimensional integrable systems. Moreover

(5.4) provides a kind of separation of variables x, t_n of the Li equation, i. e. we can separably solve the Hamiltonian systems (4.8) and (4.11) to find solutions of the Li equation, and (5.4) also provides a Bäcklund transformation among Li equation (2.5).

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